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ENCYKLOPAEDIE DER MATHEMATISCHEN WISSENSCHAFTEN.

By DR. GEORGE BRUCE HALSTED.

Mit Unterstuetzung der Akademicen der Wissenschaften zu Muenchen und Wien und der Gesellschaft der Wissenschaften zu Goettingen, herausgegeben von *H. Burkhardt* und *W. F. Meyer*. Band I. Heft 1. Leipzig, Teubner. 1898. Pages 1—112.

This is an undertaking of extraordinary importance and promise. Its aim is to give a conservative presentation of the assured results of the mathematical sciences in their present form, while, by careful and copious references to the literature, giving full indications regarding the historic development of mathematical methods since the beginning of the nineteenth century. The work begins with 27 pages on the foundations of arithmetic by Hermann Schubert of Hamburg. Schubert's reputation was made by his remarkable book on enumerative geometry. He has since applied modern ideas in an elementary arithmetic, and is known in America as a contributor to the *Monist*. Unfortunately, Schubert has made in public some strange slips. In an article "On the nature of mathematical knowledge," in the *Monist*, Vol. 6, page 295, he says: "Let me recall the controversy which has been waged in this century regarding the eleventh axiom of Euclid, that only one line can be drawn through a point parallel to another straight line. The discussion merely touched the question whether the axiom was capable of demonstration solely by means of the other propositions or whether it was not a special property, apprehensible only by sense-experience, of that space of three dimensions in which the organic world has been produced and which therefore is of all others alone within the reach of our powers of representation. The truth of the last supposition affects in no respect the correctness of the axiom but simply assigns to it, in an epistemological regard, a different status from what it would have if it were demonstrable, as was one time thought, without the aid of the senses, and solely by the other propositions of mathematics."

If Schubert had written this seventy-five years ago it might have been pardonable. Just at the beginning of this century Gauss was trying to prove this Euclidean parallel-postulate. Even up to 1824 he was in Schubert's state of mind, for he then writes Taurinus: "Ich habe daher wohl zuweilen *in Scherz* den Wunsch geaeussert, dass die Euclidische Geometrie nicht die Wahre waere." But the joke had even then gone out of the matter if Gauss had but known it, for in 1823 Bolyai Janos had written to his father, "from nothing I have created a wholly new world." Of the geometry of this world as given also by Lobachevski, Clifford wrote: "It is quite simple, merely Euclid without the vicious assumption." But this assumption is only vicious if supposed to be "apprehensible by sense-experience" or "demonstrable by the aid of the senses." That "the organic world has been produced" in Euclidean space can never be demonstrated

in any way whatsoever. On the other hand, the mechanics of actual bodies might be shown by merely approximate methods to be non-Euclidean.

Therefore Schubert's contribution on the foundations of arithmetic may fairly be read critically. He begins with counting, and defines number as the result of counting. This is in accord with the theory that their laws alone define mathematical operations, and the operations define the various kinds of number as their symbolic outcome. There is no word of the primitive number-idea, which is essentially prior to counting and necessary to explain the cause and aim of counting. This primitive number-idea is a creation of the human mind, for it only pertains to certain other creations of the human mind which I call artificial individuals. The world we consciously perceive is a mental phenomenon. Yet certain separable or distinct things or primitive individuals we cannot well help believing to subsist somehow 'in nature' as well as in conscious perception. Now by taking together certain of these permanently distinct things or natural individuals the human mind makes an artificial individual, a conceptual unity.

Number is primarily a quality of such an artificial individual. The operation of counting was made to apply to such an individual to identify it with one of a standard set of such artificial individuals, and so to get the exact shade of its numeric quality. These standard individuals were primarily sets of fingers. Then came the written standard set, *e. g.* III, or $1+1+1$; and finally the written symbol 3. Such symbols serve to represent and convey the numeric quality. The word number is applied indiscriminately to the quality or idea and to its symbol.

Schubert tells us that in antiquity the Romans represented the numbers from one to nine by rows of strokes, as 4 is still represented on our watches; while the Aztecs used to put together single circles for the numbers from one to nineteen. I have seen Japanese use columns of circles in the same way. Thus also our striking clocks convey a numeric quality by a group possessing it. But the number pertaining to a group or artificial individual is far from being the simple notion it seems. If numbers are used to express exactly this definite attribute of finite systems they are called cardinal numbers.

Schubert's first sentence is: Dinge *zaehlen* heisst, sie als gleichartig ansehen, zusammen auffassen, und ihnen einzeln andere Dinge zuordnen, die man auch als gleichartig ansieht. This may be rendered: "*To count* things means, to consider them as alike, to take them together, and to associate them singly to other things which one also considers as alike." I would prefer to say: "To count distinct things means to make of them an artificial individual or group and then to identify its elements with those of a familiar group."

When the mind of man made these artificial individuals, they were found to possess a sort of property or quality which was independent of the distinctive marks of the natural individuals composing them, also independent of the order or sub-association of these natural individuals. Whether the artificial individual were made of a church, a noise, and a pain, or made of three peas, or composed of two eyes and a nose, it had one certain quality, it was a triplet.

I see no necessity for Schubert to consider the church as like the noise and the pain. Again, the individuals of the familiar group used in the count need not be alike. Even the individuals used by a clock in counting differ ordinarily, and when we follow the count of the clock we use words all different. The primitive written number is such a picture of a group of individuals as represents their individual existence and nothing more, *e. g.* III; so however different they may be, this number is independent of the order in which they are associated with its elements.

Schubert wastes three sentences on the so-called concrete number, *benannte* Zahl. Three quails is not a number, but is a particular bevy.

His §2 *Addition*, he begins thus: "If one has two groups of units such that not only all units of each group are alike, but that also each unit of the one group is like each unit of the other group, etc." All this likeness and alikeness seems unnecessary. Any two groups may be thought into one group. Any two primitive numbers may be added.

In section 5, Peacock's Principle of Permanence is given it Hankel's general form: The combination of two numbers by any defined operation is a number, such that the combination may be handled as if it gave one of the previously defined numbers. New kinds of numbers, like all numbers, are defined by the operations from which they result. Thus are introduced zero and negative numbers, and later, the fraction. After this all is easy to the end of Schubert's contribution. It only remains to point out, as of especial importance, that from beginning to end not the slightest mention is made of measurement. Not a word is wasted on people who do not clearly see that number is long prior to measurement.

The second section of the Encyklopaedie is "Kombinatorik" by E. Netto. This is a part of mathematics which never fulfilled the hopes of the school which was lost in it during the early part of this century. Of the most comprehensive monographs the last two are in 1826 and 1837. For us it has gone over into determinants, and more than half of Netto's article is devoted to determinants. This article is particularly valuable from a bibliographic and historic point of view.

The third section is "Irrationalzahlen und Konvergenz unenlicher Prozesse," by A. Pringsheim. It begins on page 47, and goes past the end of the Heft. This is a modern subject, of intense living interest. How entirely modern it is might not be suspected by readers of such sentences in Cajori's excellent history of mathematics as those on page 70; "The first incommensurable ratio known seems to have been that of the side of a square to its diagonal, as $1:\sqrt{2}$. Theodorus of Cyrene added to this the fact that the sides of squares represented in length by $\sqrt{3}$, $\sqrt{5}$, etc., up to $\sqrt{17}$, and Theaetetus, that the sides of any square, represented by a surd, are incommensurable with the linear unit."

Now in fact Theodorus and Theaetetus made no representation whatever of the length of these sides, simply saying, *e. g.* that the side of the square whose area is 3 is incommensurable with the side of the square whose area is one. For

Euclid there was no such ratio as $1:\sqrt{2}$; for 1 is a number and so if it could have had a ratio to $\sqrt{2}$ this would have been a number. But Euclid, Book X, Proposition 7 is: "Incommensurable magnitudes are not to one another in the ratio of one number to another number." The Hindus were the first to recognize the existence of irrational numbers. Even through the middle ages and the renaissance they were absurd fictions, "numeri surdi," a designation attributed to Leonardo of Pisa. The first writer to genuinely treat them was Stifel (1544), and even he had not completely freed himself from the older terminology, since he says: "sic irrationalis numerus non est verus atque lateat sub quadam infinitatis nebula." In reference to the next step, the conceiving of ratio as number, Pringsheim says, page 51, "Hatte schon *Descartes* beliebige *Streckenverhaeltnisse* mit *einfachen Buchstaben* bezeichnet, und damit *wie mit Zahlen* gerechnet, etc." Bnt here I think the careful German has slipped.

In regard to just this point a common error is still widespread, which we see in the following, read before Sections A and B of the American Association for the Advancement of Science, 1891:

"The doctrine of Descartes was, that the algebraic symbol did not represent a concrete magnitude, but a mere number or ratio, expressing the relation of the magnitude to some unit. Hence that the product of two quantities is the product of ratios, . . . ; that the powers of a quantity are ratios like the quantity itself," etc.

That every statement here quoted is a mistake will be instantly seen from the following, taken from pages numbered 297—9 of the *original edition* of Descartes' Geometrie, 1637, a copy of which (perhaps unique on this continent) I have had the good fortune to possess since my student days (1876).

"Et comme toute l'Arithmetique n'est composée, que de quatre ou cinq operations, que sont l'Addition, la Soustraction, la Multiplication, la Diuision, & l'Extraction des racines, qu'on peut prendre pour vne espece de Diuision: Ainsi n' at' on autre chose a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a estre connues, que leur en adiouster d'autres, ou en oster; Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouuer que quatriesme. qui soit a l'vne de ces deux, comme l'autre est a l'vnité, ce qui est le mesme que la Multiplication; oubien en trouuer vne quatriesme, qui soit a l'vne de ces deux, comme l'vnité est a l'autre, ce qui est le mesme que la Diuision; au enfin trouuer vne, ou deux, ou plusieurs moyennes proportionnelles entre l'vnité, & quelque autre ligne; ce qui est le mesme que tirer la racine quarrée, ou cubique, &c. Et ie ne craindray pas d'introduire ces termes d'Arithmetique en la Goometrie, affin de me rendre plus intelligible. * * *

"Mais souuent on n'a pas besioin de tracer ainsi ces lignes sur le papier, & il suffist de les designer par quelques lettres, chascune par vne seule. Comme pour adiouster la ligne BD a GH, ie nomme l'vne a & l'autre b, & escris $a+b$; Et $a—b$, pour soustraire b d' a; Et ab, pour les multiplier l'vne par l' autre; Et $\frac{a}{b}$, pour diuiser a par b; Et aa, ou a^2, pour multiplier a par soy mesme; Et a^3, pour le multiplier encore vne fois par a, & ainsi a l' infini; Et $\sqrt{a^2+b^2}$, pour tirer la racine quarrée d' a^2+b^2; Et $\sqrt{C.a^3-b^3+abb}$, pour tirer la racine cubique d' a^3-b^3+abb, & ainsi des autres.

"Ou il est a remarquer que par a^2 ou b^3 ou semblables, ie ne concoy ordinairement que des lignes toutes simples, encore que pour me seruir des noms vsités en l'Algebre, ie les nomme des quarrés, ou des cubes, &c."

Thus what Descartes really did was to make a geometric algebra, in which, however, the product of two *sects* (Strecken) was not a rectangle but a sect; the product of three sects not a cuboid but a sect. Here Descartes represents by the single letters a, b, sects, Strecken, not *Streckenverhaeltnisse.* Descartes does not here pass beyond Euclid's representation of the ratio of two magnitudes by two other magnitudes, does not reach the conception of the systematic representation of the ratio of two magnitudes by one magnitude, that one magnitude to be always interpreted as a number. This radical innovation, the creation of this epoch-marking paradox, is due to Newton. Newton takes this vast step explicitly and consciously. The lectures which he delivered as Lucasian professor at Cambridge were published under the title "Arithmetica Universalis." At the beginning of his Arithmetica Universalis he says, page 2, "Per Numerum non tam multitudinem unitatum quam abstractam quantitatis cujusvis ad aliam ejusdem generis quantitatem quae pro unitate habetur rationem intelligimus." [In quoting this, Pringsheim, page 51, misses the first word. He omits the *Per.*] As Wolf puts it, (1710) "Number is that which is to unity as a piece of a straight line [a sect] is to a certain other sect." Thus the length of any sect is a real number, and the length of any possible sect incommensurable with the unit sect is an irrational number. Says Hayward in his Vector Algebra (1892), page 5, "Number is essentially *discrete* or *discontinuous*, proceeding from one value to the next by a finite increment or jump, and so cannot, except in the way of a limit, represent, relatively to a given unit, a continuous magnitude for which the passage from one value to another may always be conceived as a *growth* through every intermediate value." But the moment we accept Newton's definition of number it takes on whatever continuity is possessed by the sect. However, from this alone does not follow that for every irrational there is a sect whose length would give that irrational. G. Cantor was the first to bring out sharply that this is neither self-evident nor demonstrable, but involves an essential pure geometric assumption. To free the foundations of general arithmetic from such *geometric* assumption, G. Cantor and Dedekind each developed his pure arithmetic theory of the irrational. Professor Fine in his "Number-System of Algebra" seems to miss this point completely. He gives, page 42, what purports to be a demonstration that "Corresponding to every real number is a point on the line, the distance of which from the null-point is represented by the number," without any mention of the geometric assumption necessary, and then proceeds, page 43, to borrow the continuity of his number system from the näively supposed continuity of the line, the very thing for the avoidance of which G. Cantor and Dedekind made their systems. Says Dedekind, "Um so schoener erscheint es mir, dass der Mensch ohne jede Vorstellung von messbaren Groessen, und zwar durch ein endliches System einfacher Denkschritte sich sur Schoepfung des reinen, stetigen Zahlenreiches aufschwingen kann; und erst mit diesem Huelfsmittel wird es ihm nach meiner Ansicht moeglich, die Vorstellung von stetigen Raume zu einer deutlichen auszubilden."

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